Complex Notes

Winter 2023
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I August 2014

(1) Use contour integration to show that for all $a > 0$,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1} dx = \pi e^{-a}.$$

justify any limits and integrals

Proof. Consider the complex function

$$f(z) = \frac{e^{aiz}}{1+z^2}$$

notice that

$$\Re \int_{-\infty}^{\infty} f(z)dz = \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1} dx$$

so it suffices to compute $\Re \int_{-\infty}^{\infty} f(z)dz$. Consider the closed upper semicircle γ centered at the origin going from $-R$ to R along the real axis, where $R >> 1$. Then the residue theorem tells us that

$$\frac{1}{2\pi i} \int_{\gamma} f(z)dz = \sum \mathrm{Res_z f}$$

Notice that $f(z)$ has a simple pole at $z = i$ and no other poles inside the region in question for any such large value of R.

$$\mathrm{Res_i f} = \lim_{z \to i} \frac{(z-i)e^{iaz}}{(z-i)(z+i)} = \frac{e^{-a}}{2i}$$

It remains now to show that the contribution of the upper arc α of the semicircle parameterized by $z(\theta) = Re^{i\theta}$ with $\theta \in [0, \pi]$ contributes no part to the integral. Indeed,

$$\begin{aligned} \left| \int_{\alpha} f(z) \right| dz &= \int_0^{\pi} \frac{e^{iaRe^{i\theta}} Rie^{i\theta}}{1+R^2e^{2i\theta}} d\theta \\ &\leq \sup_{\theta \in [0,\theta]} \frac{Re^{-aR\sin\theta}}{R^2-1} \end{aligned}$$

as $R \to \infty$, it is clear that since $a > 0$ that $Re^{-aR\sin\theta}R^2 - 1 \to 0$ for any value of $\theta \in [0, \theta]$ (since $\sin(\theta) \geq 0$ for $\theta \in [0, \pi]$).

Thus in the limit as $R \to \infty$, $\int_{\alpha} f(z)dz \to 0$ so

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i e^{-a}/2i = \pi e^{-a}$$

Taking the real part we see that

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1} dx = \pi e^{-a}$$

as required.

□

(2) Let $f(x)$ be a continuously differentiable real-valued function over $(-\infty, \infty)$ with $f(0) = 0$. Suppose that $|f'(x)| \leq |f(x)|$ for all $x \in (-\infty, \infty)$.

(a) Show that $f(x) = 0$ for all x in a neighborhood $(-\epsilon, \epsilon)$ of 0, for some $\epsilon > 0$.

Proof. Consider the interval $[0, 1/2]$. Since this interval is compact and f is continuous, f attains a maximum on this interval, call it M. Then notice that $M \geq |f(x)| \geq |f'(x)|$ for all $x \in [0, 1/2]$. by application of the mean value theorem we see that

$$|f(x)| = |f(x) - f(0)| = |f'(c)(x - 0)| \leq Mx \leq M/2$$

from which we see that $M/2 \leq M$ so M must be zero. By the same process we see that on $[-1/2, 0]$, $f(x) = 0$. □

(b) Show that $f(x) = 0$ for all $x \in (-\infty, \infty)$.

Proof. We can show this inductively on succesive intervals $[n/2, (n + 1)/2]$. Since the base case was done in the part (a) assume that $f(x) = 0$ on $[n/2, (n+1)/2]$, then we know that $f((n+1)/2) = 0$ and by compactness f attains a maximum M on the interval $[(n + 1)/2, (n + 2)/2]$. From this we see that $M \geq |f(x)| \geq |f'(x)|$ for $x \in [(n + 1)/2, (n + 2)/2]$ thus by the MVT,

$$|f(x) - 0| = |f'(c)(x - (n + 1)/2| \leq M(x - (n + 1)/2) \leq M/2$$

and as before we see that $M = 0$ so $f(x) = 0$ on $[(n + 1)/2, (n + 2)/2]$. This $f(x) = 0$ for all $x \in [0, \infty)$. A similar argument shows that $f(x) = 0$ for $(-\infty, 0]$. □

(3) Let $D_1 \subset \mathbb{C}$ be the open disc centered at i with radius 1, and let $D_2 \subset \mathbb{C}$ be the open disc centered at $3/2i$ with radius $1/2$. Find an explicit biholomorphic map sending $\Omega = D_1 - D_2$ onto the open unit disc in $\mathbb{C}$. You may express this solution as a composition of biholomorphic maps so long as each of those maps is written explicitly.

Proof. First apply a rotation $e^{-\pi i/2}$. Then notice that the map $F(z) = \frac{z-i}{z+i}$ which is the Cayley transform that maps conformally $\mathbb{H}$ to $\mathbb{D}$ will map the line $\{x + i \mid x \in \mathbb{R}\}$ to the circle centered at $1/2$ with radius $1/2$. A simple computation verifies that $F(2i) = 1/3$ lies inside this circle and so everything in the strip $S = \{x + iy \mid y \in (0, 1)\}$ is mapped into $e^{-\pi i/2}(\Omega)$. Since F is conformal restrcited to S, apply the inverse $G : e^{\pi i/2}(\Omega) \to S$. Then we apply $\log(\pi z)$ which is a conformal map from S to $\mathbb{H}$. Finally we apply the map $F : \mathbb{H} \to \mathbb{D}$. The explicit map then will be

$$F \circ \log(\pi \cdot) \circ G \circ e^{-\pi i/2} : \Omega \to \mathbb{D}$$

as required. □

II January 2015

III January 2016

(4) Suppose f is an entire function with $\int\int_{\mathbb{C}} |f(z)|^2 dxdy < \infty$. Show that $f(z) = 0$ for all $z \in \mathbb{C}$.

Proof. Let $g(z) = f^2(z)$. Since $|g(z)| = |f(z)|^2$, and g is holomorphic on $\mathbb{C}$, it suffices to show that if $\int\int_{\mathbb{C}} |g(z)|dxdy < \infty$, then g is zero. Recall the mean value theorem which tells us that

$$g(0) = \frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta})d\theta$$

for any $r > 0$ since g is entire. Then multiply both sides by r and integrate from 0 to R

$$\begin{aligned}\int_0^R g(0)rdr &= \frac{1}{2\pi}\int_0^R\int_0^{2\pi} g(re^{i\theta})rd\theta dr \\ g(0) &= \frac{1}{R^2\pi}\int_0^R\int_0^{2\pi} g(re^{i\theta})rd\theta dr \\ &= \frac{1}{\pi R^2}\iint_{\mathbb{D}_R(0)} g(z)dxdy\end{aligned}$$

then $|g(0)| \leq \frac{1}{\pi R^2}\iint_{\mathbb{D}_R(0)}|g(z)|dxdy$ and in the limit as $R \to \infty$, $\iint_{\mathbb{D}_R(0)}|g(z)|dxdy \to \iint_{\mathbb{C}}|g(z)|dxdy < \infty$ and so $|g(0)| \leq \frac{1}{\pi R^2}\iint_{\mathbb{C}}|g(z)|dxdy \to 0$. Thus $|g(0)| = 0$. We can repeat this procedure to evaluate $|g(z)|$ for any $z \in \mathbb{C}$ to show that $|g(z)| = 0$. Thus $g(z) = 0$. □

IV August 2017

(3)

(A) Let $B = \{(x, y) \mid x^2 + y^2 < 1\}$ and let $u(x, y)$ be a harmonic function defined on some open set U containing the closure of B. Prove that

$$u(0,0) = \frac{1}{\pi}\int_B u(x,y)dxdy$$

Proof. Notice that the statement implies that $u(x, y)$ is harmonic in an open ball of radius $R > 1$ containing B, then the mean value theorem of harmonic functions which states that

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\theta})d\theta$$

for any $0 < \rho \leq 1 < R$. Multiplying by ρ and integrating both sides from 0 to 1

$$\begin{aligned}\int u(0)\rho d\rho &= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} u(\rho e^{i\theta})\rho d\theta d\rho \\ \frac{u(0)}{2} &= \frac{1}{2\pi}\int_B u(x,y)dxdy\end{aligned}$$

as required. □

(B)

V January 2018

(1) Let $\lambda > 1$ be a real number. Show that the equation $ze^{\lambda - z} = 1$ has a real solution in the unit disk, and that there are no other solutions in the unit disk.

Proof. First let us notice that $e^{\lambda-z} \neq 0$ for all z thus $ze^{\lambda-z} = 1$ if and only if $z = e^{z-\lambda}$. Now the function $f : [-1, 1] \to \mathbb{R}$, $f(x) = x - e^{x-\lambda}$ has the following properties: $f(-1) = -1 - e^{-(x+\lambda)} < 0$ and $f(1) = 1 - e^{1-\lambda} > 0$ since $\lambda > 1$, thus by the intermediate value theorem there exists $c \in [-1, 1]$ such that $f(x) = 0$.

To show that there are no other solutions in the unit disk, let us notice that for all $z \in \overline{\mathbb{D}}$

$$|e^{z-\lambda}| \leq e^{\Re(z)-\lambda} < 1 \qquad \text{since } \lambda > 1 \text{ and } \Re(z) < 1$$

moreover $|z| = 1$ for all z on the unit circle so by Rouche's theorem z and $z - e^{\lambda - z}$ have the same number of zeros in the unit circle $\mathbb{D}$. Since z has a unique zero at the origin, this tells us that the real solution we found in the first part of this problem is the only solution in the unit disk. $\square$

(2) Let $\gamma(t) : [0, b] \to \mathbb{C}$ be a piecewise differentiable smooth function describing a curve Γ in the complex plane

(A) For $a \neq \Gamma$, let

$$h(u) = \int_0^u \frac{\gamma'(t)}{\gamma(t) - a} dt$$

Differentiate $e^{-h(u)}(\gamma(u) - a)$ and prove that $e^{h(u)} = \frac{h(u)-a}{h(0)-a}$ for all $0 \leq u \leq b$.

Proof. First

$$\frac{d}{du} e^{-h(u)}(\gamma(u) - a) = -h'(u)e^{h(u)}(\gamma(u) - a) + e^{-h(u)}\gamma'(u)$$

and notice by the fundamental theorem of calculus that $h'(u) = \frac{d}{du}\int_0^u \frac{\gamma'(t)}{\gamma(t)-a} dt = \frac{\gamma'(u)}{\gamma(u)-a}$ so

$$\begin{aligned} &= -\frac{\gamma'(u)}{\gamma(u) - a} e^{h(u)}(\gamma(u) - a) + e^{-h(u)}\gamma'(u) \\ &= (\gamma'(u) - \gamma'(u))e^{h(u)} = 0 \end{aligned}$$

thus $e^{-h(u)}(\gamma(u) - a) = k$ for some constant k. To determine this constant notice that $e^{-h(0)}(\gamma(0) - a) = \gamma(0) - a$ and so we see that

$$e^{h(u)} = \frac{\gamma(u) - a}{\gamma(0) - a}$$

as required. $\square$

(B) Use (A) to show that if Γ is a closed path then $\int_\Gamma (z - a)^{-1} dz$ is an integer multiple of $2\pi i$. Show that this integral is zero if Γ is contained in the interior of a disk not containing a.

Proof.

$$\int_\Gamma \frac{1}{z - a} dz = \int_0^b \frac{\gamma'(t)}{\gamma(t) - a} dt = h(b)$$

Notice then by part (A) that

$$e^{h(b)} = \frac{\gamma(b) - a}{\gamma(0) - a} = 1$$

since $\gamma(b) = \gamma(0)$ as Γ is a closed curve. We then see that $h(b) = n2\pi i$ for $n \in \mathbb{Z}$. The last conclusion follows from the fact that $1/(z - a)$ is holomorphic for all $z \neq a$ and so by Cauchy's theorem must vanish if Γ is contained in a closed disk not containing a. $\square$

(4) Define $D = \{z \in \mathbb{C} \mid 2 < |z| < 3\}$. Let f be a holomorphic function over D that is continuous over $\overline{D}$.

(A) Suppose that $\max_{|z|=2} |f(z)| \leq 2$ and $\max_{|z|=3} |f(z)| \leq 3$. Prove that $|f(z)| \leq |z|$ on D.

Proof. Consider the map $g(z) = \frac{f(z)}{z}$. Notice that since $f(z)$ is nonvanishing, that $g(z)$ is holomorphic on D. Thus for all $|z| = 2$,

$$|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{2}{2} = 1$$

and similarly when $|z| = 3$,

$$|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{3}{3} = 1$$

so we then apply the maximum-modulus principle to $g(z)$, where we see that $|g(z)| \leq 1$ for all $z \in D$. Thus

$$|f(z)| \leq |z|$$

for all $z \in D$. □

(B) Suppose that $|f(z)| = |z|$ for $|z| = 2$ and $|z| = 3$. Suppose furthermore that $f(z)$ does not have any zeros in D. Prove that $f(z) = e^{i\theta} z$ for some constant $\theta \in [0, \pi]$.

Proof. Let us define the following function

$$h(z) = \ln |f(z)| - \ln |z|$$

and notice that this function is harmonic on D. Furthermore, we see that when $|z| = 2$, that $|h(z)| = \ln |2| - \ln |2| = 0$ and similarly when $|z| = 3$ that $|h(z)| = \ln |3| - \ln |3| = 0$. We conclude then by the maximum-modulus principle for harmonic functions that $h(z) = 0$ on D, thus

$$\ln |f(z)| = \ln |z|$$

and so $|f(z)| = |z|$. Thus $f(z) = e^{i\theta} z$ as required. □

VI August 2018

(1) Let U be a connected domain in $\mathbb{C}$.

(A) Let $h(z)$ be harmonic on U, and $f : U \to U$ be a holomorphic function. Prove that $h \circ f$ is a harmonic function on U.

Proof. Recall that $\Delta = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}}$. Now applying the complex chain rule,

$$\frac{\partial (h \circ f)}{\partial \overline{z}} = \frac{\partial h}{\partial z} \frac{\partial f}{\partial \overline{z}} + \frac{\partial h}{\partial \overline{z}} \frac{\partial \overline{f}}{\partial \overline{z}}$$

since f is holomorphic the first term disappears, so $\frac{\partial (h \circ f)}{\partial \overline{z}} = \frac{\partial h}{\partial \overline{z}} \frac{\partial \overline{f}}{\partial \overline{z}}$. Now

$$\begin{aligned} \frac{\partial}{\partial z} \frac{\partial (h \circ f)}{\partial \overline{z}} &= \frac{\partial}{\partial z} \left(\frac{\partial h}{\partial \overline{z}} \frac{\partial \overline{f}}{\partial \overline{z}} \right) \\ &= \left(\frac{\partial}{\partial z} \frac{\partial h}{\partial \overline{z}} \right) \frac{\partial \overline{f}}{\partial \overline{z}} + \frac{\partial h}{\partial \overline{z}} \left(\frac{\partial}{\partial z} \frac{\partial \overline{f}}{\partial \overline{z}} \right) \end{aligned}$$

The first term is zero since h is harmonic, and the second term is zero since $\overline{f}$ is antiholomorphic and therefore

$$\frac{\partial}{\partial z} \frac{\partial \overline{f}}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}} \frac{\partial \overline{f}}{\partial z} = 0$$

thus $h \circ f$ is harmonic. □

(B) Let $h(z)$ be a real valued harmonic function on U such that $(h(z))^2$ is also a harmonic function on U. Prove that $h(z)$ must be constant.

Proof. First let us define $\widetilde{h}(z) = h(z) - h(w)$ for some fixed $w \in U$. Now notice that $\tilde{h}$ is harmonic and $\tilde{h}^2$ is also harmonic and remains real valued. Now since the image of $\tilde{h}$ is entirely real, it follows that the image of $-\tilde{h}^2$ is contained in $\mathbb{R}_{\leq 0}$. Now furthermore notice that $-\tilde{h}^2(w) = 0$ is a maximum of the function on U, which by the maximum modulus principle for harmonic functions implies that $\tilde{h}^2(z) = 0$ for all z and hence $h(z) = h(w)$ is constant. □

(2) Let $z_1 \neq z_2 \in \mathbb{C}$.

(A) Construct all Biholomorphic maps of the complex plane which have z_1 and z_2 as their fixed points.

Proof. Biholomorphic maps on $\mathbb{C}$ are contained in the set of injective entire functions, but this set consists of only the linear maps $f(z) = az + b$ with $a \neq 0$. To see this notice that $f(1/z)$ cannot be a removable singularity since otherwise f would be bounded and hence constant by Liouville's theorem. $f(1/z)$ cannot be an esential singularity since the Casorati-Weierstrass theorem implies that for any open set containing 0, $f(1/z)$ is dense, thus implying that f is not injective. Lastly, we see that f has a pole at infinity. Since f has no other poles, it follows that $f(z) = (az + b)^n$. In order for f to remain injective, $n = 1$, thus $f(z) = az + b$, $a \neq 0$. Since linear maps of the form $az + b$, $a \neq 0$, are surjective, it follows that these are all of the biholomorphisms of $\mathbb{C}$. Now if $z_1 = az_1 + b$ and $z_2 = az_2 + b$. If $a \neq 1$, then

$$z_1 = \frac{b}{1-a} = z_2$$

which is a contradiction so $a = 1$, then $z_1 = z_1 + b$ implying that $b = 0$ so the only biholomorphism of $\mathbb{C}$ fixing both z_1 and z_2 is the identity map $f(z) = z$. □

(B) Construct all biholomorphisms of $\widehat{\mathbb{C}}$ such that z_1 and z_2.

Sketch of Proof

Recalling that the automorphisms of $\mathbb{C} \cup \{\infty\}$ are $\{\frac{az+b}{cz+d} \mid a, b, c, z \in \mathbb{C}, ad - bc = 1\}$. Now in order to fix z_1 and z_2 we obtain the following additional linear constraints.

$$cz_2^2 + dz_2 - az_2 - b = 0$$
$$cz_1^2 + dz_1 - az_1 - b = 0$$

All together we have three independent constraints and 4 unknowns which implies that the solution space is one dimensional. It is easy to find biholomorhpisms which send z_1 to 0 and z_2 to ∞, and then construct its inverse. Let A, A^{-1} denote such a transformation. We then see that $A^{-1} \circ e^{i\theta} \circ A : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ are all conformal maps fixing z_1 and z_2 for any value of θ. These are all of them as we have a 1-dimensional space worth of solutions.

VII August 2020

(5) Let f be holomorphic on a neighborhood of the closed unit disk centered at the origin. Assume that $|f(z)| = 1$ on $|z| = 1$, , and is not a constant on the disc. Prove that there exist a positive integer k, points $\alpha_1, \ldots, \alpha_k$, in the open unit disk positive integers $m_1, \ldots, m_k$, and positive integers $m_1, \ldots, m_k$, and a complex number β such that

$$f(z) = \beta \prod_{i=1}^{k} \left(\frac{z - \alpha_i}{1 - \overline{\alpha_i}} \right)^{m_i}$$

for all $z \in \mathbb{D}$.

Proof. Since $f(z)$ is analytic in an open neighborhood of the disk, $|f(z)| \to 1$ uniformly as $|z| \to 1$, thus there is some $r > 1$ such that $|f(z)| > 0$ for all $z \in \{z \mid r < |z| < 1\}$. It follows then that f can only have a finite number of zeros in the disk as otherwise the set of zeros would contain a limit point (by Bolzano-Weierstrass) and by uniqueness of analytic continuuation would be the zero function. Let $\alpha_1, \ldots, \alpha_k$ be those zeros and let $m_1, \ldots, m_k$ be the multiplicities of those zeros, then

$$g(z) = \prod_{i=1}^{k} \left(\frac{z - \alpha_i}{1 - \overline{\alpha_i}} \right)^{m_i}$$

is an analytic function on the disk such that $|f(z)|1$ on $|z| = 1$. We then want to show that all such g with those prescribed zeros and multiplicities is equivalent to $g(z)$ up to rotation. Let $h(z)$ be another function which has the same zeros and multiplicities, then $h(z)/g(z)$ and $g(z)/h(z)$ are both analytic functions on the disk with norm one on $|z| = 1$, thus maximum modulus tells us that $|h(z)| \leq |g(z)|$ and $|g(z)| \leq |h(z)|$, so $|g(z)| = |h(z)|$ and we see that $h(z) = \beta g(z)$ for some $|\beta| = 1$. ☐

VIII January 2021

(1) Prove that all 5 roots of $2z^5 + 8z - 1$ lie in the disk $|z| \leq 2$ but only one root lies inside $|z| < 1$.

Proof. First let us see that for $z \in \{|z| = 1\}$, $|2z^5 - 1| \leq 2|z|^5 + 1 = 3$ and $|8z| = 8$ for all $z \in \{|z| = 1\}$, thus by Rouche's theorem, $8z$ and $2z^5 + 8z - 1$ have the same number of zeros inside $|z| < 1$. Since $8z = 0$ when $z = 0$ is the only zero in $|z| < 1$, it follows that $2z^5 + 8z - 1$ has one zero inside $|z| < 1$.

Now on $|z| = 2$, $|8z - 1| \leq 8|z| + 1 = 17$ and $|2z^5| = 2|z|^5 = 2^6 = 64$ so by Rouche's theorem $2z^5$ and $2z^5 + 8z - 1$ have the same number of zeros in $|z| < 2$. Since $2z^5 = 0$ for all 5 roots of unit which have modulus 1 it follows that $2z^5 + 8z - 1$ has 5 roots inside $|z| < 2$. ☐

(2) Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function which satisfies:

$$|f(z)| \leq 1 \text{ and } f(i) = 0$$

Prove that for all $z \in \mathbb{H}$,

$$|f(z)| \leq \frac{|z - i|}{|z + i|}$$

Proof. First let us recall that $G(z) = i\frac{i-z}{i+z}$ is a biholomorphism $\mathbb{D} \to \mathbb{H}$ and $G(0) = i$. We then notice that $f \circ G : \mathbb{D} \to \mathbb{D}$ and $f(G(0)) = 0$. It follows from Schwarz's lemma that $|f \circ G(z)| \leq |z|$. To complete the proof, recall that the inverse biholomorphism to G is the map $F(z) = \frac{i-z}{i+z}$, so replaciing z with $F(z)$ we see that

$$|f \circ G(F(z))| \leq |F(z)|$$
$$|f(z)| \leq \frac{|z - i|}{|z + i|}$$

as required. ☐

(5) Let R be the parallelogram with vertices $(0, 0)$, $(1, 1)$, $(3, 0)$, and $(2, -1)$. Evaluate the integral

$$\iint_R (x + 2y)^2 e^{x-y} dA$$

Proof. Let us begin by doing a change of basis so that the vectors in the direction of the sides of the parallelogram become our new basis vectors. Indeed we see that $x = y = 0$ and $-1/2x - y = x + 2y = 0$ are the expressions for 2 of our lines and then $x - y = 3$ and $x + 2y = 3$ are the expressions for the second two. We then let $u = x - y$ and $v = x + 2y$ and the bounds of integration change to 0 to 3 in both variables. We now need to compute the determinant of the Jacobian: notice $y = \frac{1}{3}(v - u)$ and $x = \frac{1}{3}(2u + v)$ so that

$$\det \begin{pmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{pmatrix} = 1/3$$

then our integral becomes

$$\begin{aligned} \iint_R (x + 2y)^2 e^{x-y} dA &= \int_0^3 \int_0^3 v^2 e^u \frac{1}{3} du dv \\ &= \int_0^3 v^2(e^3 - 1)\frac{1}{3} dv \\ &= 3(e^3 - 1) \end{aligned}$$

□

IX Fall 2021

(2) Use calculus of residues to explicitly compute $\int_0^\infty \frac{x^n}{1+x^{2n}} dx$. Here $n \geq 2$ is a positive integer.

Proof. Let us consider the following complex function $f(z) = \frac{z^n}{1+z^{2n}}$. Our goal will be to integrate $f(z)$ over the sector going from 0 to R along the real axis and then across the arc α parameterized by $Re^{i\theta}$ with $\theta \in [0, \pi/n]$. Then we go back to the origin along the line L parameterized by $te^{\pi i/n}$ with $t \in [R, 0]$. Now notice that on α,

$$\left| \int_\alpha f(z) dz \right| \leq \frac{R^n}{R^{2n} - 1} \to 0$$

as $R \to \infty$ so $\int_\alpha f(z)dz = 0$ in the limit. Now we seek to compute the integral over L.

$$\begin{aligned} \int_L f(z) dz &= -\int_0^R \frac{t^n e^{\pi i}}{1 + t^{2n} e^{2\pi i}} e^{\pi i/n} dt \\ &= e^{\pi i/n} \int_0^R \frac{t^n}{1 + t^{2n}} dt \end{aligned}$$

Thus in the limit as $R \to \infty$, we see that $e^{\pi i n} \int_0^\infty \frac{x^n}{1+x^{2n}} dx = \int_L f(z)dz$.

Next we need to check if f has any poles in the interior of the contour γ. Indeed we see that at $e^{\pi i/2n}$, $(e^{\pi i/2n})^{2n} - 1 = e^{2\pi i} - 1 = 0$, so f has a pole at $e^{\pi i/2n}$, and that this is the only point in the interior of γ for which $f(z)$ has a pole. Let us evaluate the residue at this pole (and notice additionally that this pole has order 1.

$$\text{Res}_{e^{\pi i/2n}} = \lim_{z \to e^{\pi i/2n}} \frac{(z - e^{\pi i/2n}) z^n}{1 + z^{2n}} = A$$

To compute A, we can apply L'Hopitals rule since the numerator and denominator converge to 0. Then we can compute

$$\begin{aligned} \int_0^\infty f(z) dz + \int_L f(z) dz &= 2\pi i A \\ \int_0^\infty f(z) dz + e^{\pi i/n} \int_0^\infty f(z) dz &= 2\pi i A \\ \int_0^\infty f(z) dz &= \frac{2\pi i A}{1 + e^{\pi i/n}} \end{aligned}$$

□

(3) Let $D_0 = \{z \in \mathbb{C} \mid |z| < 1\}$. $f : D_0 \to \mathbb{C}$ is holomorphic on D_0 and satisfies $|f(z)| \leq \log(1/|z|)$ for any $z \in D_0$. Prove that $f \equiv 0$.

Proof. We see that $e^{|f(z)|} \leq 1/|z|$ for all $z \in D_0$. We then notice that as $|z| \to 1$, that $|f(z)| \to 0$, however the maximum modulus principle then tells us that f must be zero on the entire disk D_0. □

(5) Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$. $f : D \to \mathbb{C}$ is holomorphic, injective, and satisfies $f'(0) = 1$. Prove that the area $f(D)$ is at least π.

Proof. Green's theorem tell us that we can measure the area of a subset Ω of the plane by

$$\int_{\partial\Omega} xdy = -\int_{\partial\Omega} ydx$$

thus

$$\iint_{\Omega} dA = \frac{1}{2}\left(-\int_{\partial\Omega} iydx + i\int_{\partial\Omega} ydx\right) = \frac{1}{2i}\int_{\partial\Omega} \overline{z}dz$$

where $z = x + iy$. Since f is conformal onto its image, we can make a change of variables to see that

$$\text{Area}(f(D)) = \frac{1}{2i}\int_{\partial D} \overline{f(z)}f'(z)dz$$

from here we can expand in a power series

$$\begin{aligned}
\frac{1}{2i}\int_{D} \overline{f(z)}f'(z)dz = \frac{1}{2i}\int_{\partial D}\left(\sum_{n=1}^{\infty}\overline{a_n}z^n\right)\left(\sum_{m=1}^{\infty}na_nz^n\right)dz \\
= \frac{1}{2}\int_0^{2\pi}\sum_{k=1}^{\infty}\overline{a_k}a_kr^kd\theta + \frac{1}{2}\sum_{m\neq n}\int_{\partial D}\overline{a_n}a_me^{(m-n)i\theta}r^sd\theta \\
= \pi\sum_{k=1}^{\infty}|a_k|^2r^k + 0
\end{aligned}$$

since $\int_0^{2\pi} e^{(m-n)\pi\theta}d\theta = 0$ when $m \neq n$.

$$= \pi r + \pi\sum_{k=2}^{\infty}|a_k|^2r^k$$

now taking the limit as $r \to 1$, we obtain

$$= \pi + \pi\sum_{k=2}^{\infty}|a_k|^2$$

and we are done. □

X August 2022

(1) Show that $\ln(x^2 + y^2)$ is a harmonic function in $\mathbb{C}\backslash 0$. Find a conjugate harmonic function of $u(x, y)$ in $\mathbb{C}\backslash\{x \mid x \leq 0\}$. Show that it does not have a conjugate harmonic function in $\mathbb{C}\backslash\{0\}$.

Proof. Indeed we see that

$$\frac{\partial^2}{\partial x^2}\ln(x^2+y^2) = \frac{2(x^2+y^2)-4x^2}{(x^2+y^2)^2} \qquad \frac{\partial^2}{\partial y^2}\ln(x^2+y^2) = \frac{2(x^2+y^2)-4y^2}{(x^2+y^2)^2}$$

so $\Delta \ln(x^2+y^2) = \frac{2(x^2+y^2)-4y^2}{(x^2+y^2)^2} + \frac{2(x^2+y^2)-4x^2}{(x^2+y^2)^2} = 0$ and we conclude that $\ln(x^2+y^2)$ for all $(x,y) \neq (0,0)$. Notice that in polar coordinates $\ln(x^2+y^2) = \ln(R^2) = \ln|z|$. Taking a branch of the logarithm as prescribed by the exercise gives us that

$$\log(z) = \log|z| + i\theta$$

and we see that the conjugate harmonic function is simply arg(z).

Recall that the conjugate harmonic is unique up to a constant, but this implies then that $\log(z)$ would be holomorphic on $\mathbb{C}\backslash\{z_0\}$, but this is not true. □

(2) Evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}dx$$

Proof. Let us consider the following function $f(z) = \frac{z^2}{1+z^4}$. Consider the integral $\int_\gamma f(z)dz$ where γ is the closed upper semicircle of radius $R > 10$ going first from $-R$ to R and then along the upper arc α parameterized by $Re^{i\theta}$ with $\theta \in [0,\pi]$.

Notice that

$$\left|\int_\alpha \frac{z^2}{1+z^4}dz\right| \leq \pi R\frac{R^2}{R^4-1} \to 0 \quad \text{as } R \to \infty$$

so in the limit as $R \to \infty$, $\int_\mathbb{R} \frac{x^2}{1+x^4}dx = \int_\gamma f(z)dz$. Notice that f has two simple pole contained in γ. Let us compute the residues:

$$\mathrm{Res}_{\mathrm{e^{i\pi/4}}}\mathrm{f} = \frac{e^{i\pi/2}}{}$$

(the reader can finish this computation) Then we see that

$$\int_\mathbb{R} \frac{x^2}{1+x^4}dx = 2\pi i(A+B)$$

□

(3)

XI January 2023

(1) Let $a, b > 0$, $a \neq b$. Find $\int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+a^2)(x^2+b^2)}dx$ by using residue calculus.

Proof. First notice that $\int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+a^2)(x^2+b^2)}dx = \Re\int_{-\infty}^{\infty} f(z)dz$ where $f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$. Notice that $f(z)$ has two poles in the upper half plane at ia and ib. Let $R > \max\{a,b\}$. We want to compute the integral $\int_\gamma f(z)dz$ where γ is the semicircle starting at $-R$ going to R and then via the upper semicircle α

parameterized by $Re^{i\theta}$, $\theta \in [0, \pi]$. Now let us compute the integral of α,

$$\begin{aligned}\left|\int_\alpha f(z)\right| &= \left|\int_0^\pi \frac{e^{iRe^{i\theta}}}{(R^2e^{2i\theta}+a^2)(R^2e^{2i\theta}+b^2} iRe^{i\theta}d\theta\right| \\ &\le \pi R \sup_{\theta\in[0,\pi]} \frac{e^{-R\sin(\theta)}}{R^4+p(R)}, \quad \deg p < 4 \\ &\to 0 \quad \text{as } R\to\infty\end{aligned}$$

therefore in the limit $\int_\alpha f(z)dz \to 0$ as $R \to \infty$ and so

$$\int_{-\infty}^{\infty} f(z)dz = 2\pi i(\mathrm{Res_{ia}f} + \mathrm{Res_{ib}f}$$

Computing both of the resides leaves us with

$$\mathrm{Res_{ia}f} = \frac{\mathrm{e^{-a}}}{\mathrm{2ai(b^2-a^2)}}, \qquad \mathrm{Res_{ib}f} = \frac{\mathrm{e^{-b}}}{\mathrm{2bi(a^2-b^2)}}$$

thus

$$\int_{-\infty}^{\infty} f(z)dz = \pi\left[\frac{e^{-a}}{a(b^2-a^2)} + \frac{e^{-b}}{b(a^2-b^2)}\right]$$

which is real so we conclude that

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+a^2)(x^2+b^2)}dx = \pi\left[\frac{e^{-a}}{a(b^2-a^2)} + \frac{e^{-b}}{b(a^2-b^2)}\right]$$

□

(2) λ is purely imaginary. Prove that $z = \lambda - \frac{1}{3}e^{z^2}$ has exactly one solution in the strip $S = \{x+iy \mid |x| \le 1\}$.

Proof. We want a solution to $0 = z - \lambda - \frac{1}{3}e^{z^2}$.

Notice that

$$|\frac{1}{3}e^{z^2}| = \frac{1}{3}e^{x^2}e^{-y^2} \le \frac{e}{3}e^{-y^2} < e^{-y^2} \le 1$$

since $y^2 \ge 0$. Now notice that on the rectange centered at $z-\lambda$, which goes from $(-1, -R-\lambda)$ to $(1, -R-\lambda)$ to $(1, R-\lambda)$ to $(-1, R-\lambda)$ for all $R > 10$, $|z-\lambda| > 1$, thus by Rouche's theorem, on the interior of the rectange $\mathrm{Rect_R}$, $z-\lambda$ and $z-\lambda+\frac{1}{3}e^{z^2}$ have the same number of solutions. For all R sufficiently large, we see that $z-\lambda$ has exactly one solution at $z = \lambda \in \mathbb{S}$, thus the expression $z = \lambda - \frac{1}{3}e^{z^2}$ has exactly one solution in $\mathbb{S}$. □

(3) Let $\mathbb{D} = \{z \mid |z| < 1\}$ and $\mathbb{A} = \{z \in \mathbb{C} \mid 0 < \arg z < \frac{2\pi}{5}\}$. Find an explicit biholomorphic map $f : \mathbb{D} \to \mathbb{A}$.

Proof. First recall that the map $G(z) = i\frac{z-1}{z+1}$ is a biholomorphism from $\mathbb{D}$ to the upper half plane $\mathbb{H}$. Now recall that the map $g_\alpha(z) = z^\alpha$, $0 < \alpha < 2$, defined on the branch cut where the positive real axis is deleted is a biholomorphism from $\mathbb{H}$ to the sector $S = \{z \mid 0 < \arg z < \alpha\pi\}$. Now select $\alpha = 2/5$ and we see that the composition $g_{2/g} \circ G$ is a biholomorphism from the unit disk to the sector $\mathbb{A}$. □

(4) Let $\mathbb{S} = \{x+iy \mid -1 \le x \le 1\}$. $f : \mathbb{S} \to \mathbb{C}$ is bounded continuous function that is holomorphic on the interior of $\mathbb{S}$. For $-1 \le\le x \le 1$ let $M(x) = \sup_{y\in\mathbb{R}} |f(x+iy)|$.

(A) Suppose $M(1), M(-1) \leq 1$. Prove that $|f(z)| \leq 1$ for any $z \in \mathbb{S}$.

Proof. Consider the function

$$f_\varepsilon = \frac{1}{1+\varepsilon z} f(z)$$

We see that as $\varepsilon \to 0$, $f_\varepsilon \to f$ uniformly on $\mathbb{S}$. Notice also that for any x, as $y \to \infty$, $|f_\varepsilon(x+iy)| \leq M \left|\frac{1}{1+\varepsilon z}\right| \to 0$, where $M = \sup_{z\in\mathbb{S}} |f_\varepsilon(z)|$.

Let $R > 0$ be large enough so that for $y \geq R$, $|f_\varepsilon(x+iy)| < 1$. We can then apply the maximum modulus principle to $\mathbb{S} \cap \{x + iy \mid |y| \leq R\}$. We see then that f_ε can only obtain its maximum on the boundary and by the decay that f_ε possesses in y, we see that the maximum will occur on $x = -1$ or $x = 1$. Now $\varepsilon \to 0$ we see that f attains its maximum on $x = -1$ or $x = 1$ and therefore $\sup_{z\in\mathbb{S}} |f(z)| = \max\{M(1), M(-1)\} \leq 1$. □

XII August 2023

(1) Use Green's theorem to evaluate the integral

$$\int_C \sqrt{1+e^{x^2}}dx + 4xydy$$

where C is the boundaary of the triangle with vertices $(0,0)$, $(1,0)$, $(1,3)$.

Proof. Recall Green's theorem:

$$\int_C Pdx + Qdy = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$

Notice that

$$\frac{\partial P}{\partial y} = 0 \qquad \frac{\partial Q}{\partial x} = 4y$$

Then by Green's theorem

$$\begin{aligned}\int_C \sqrt{1+e^{x^2}}dx + 4xydy &= \iint 4ydA \\ &= \int_0^1 \int_0^{3x} 4ydydx \\ &= \int_0^1 18x^2 dx \\ &= 6\end{aligned}$$

□

(2) Assume $\xi > 0$ and compute

$$\int_{\mathbb{R}} \frac{\cos(2\pi x\xi)}{x^2+1} dx$$

Proof. First let us consider the complex function $f(z) = \frac{e^{2\pi x\xi i}}{z^2+1}$ and notice that $\Re(\int_{\mathbb{R}} f(z)dz) = \int_{\mathbb{R}} \frac{\cos(2\pi x\xi)}{x^2+1}dx$ as $e^{2\pi ix\xi} = \cos(2\pi x\xi) + i\sin(2\pi x\xi)$. Now consider the closed upper semicircle centered at the origin going from $-R$ to R along the real axis and then traveling along the arc $Re^{i\theta}$ with $\theta \in [0,\pi]$. Call the contour γ and the upper arc of the semicircle α then notice

$$\int_\gamma f(z) = \int_{\mathbb{R}} f(z)dz + \int_\alpha f(z)dz$$

and by the residue theorem

$$\int_\gamma f(z) = 2\pi i \sum_z \mathrm{Res_f(z)}$$

Notice that the only pole in the interior of the curve γ is a simple pole located at i and possesses the residue

$$\mathrm{Res_f(i)} = \lim_{\mathrm{z \to i}} \frac{\mathrm{(z-i)e^{2\pi iz\xi}}}{\mathrm{(z-i)(z+i)}} = \frac{\mathrm{e^{-2\pi\xi}}}{\mathrm{2i}}$$

Now let us integrate $\int_\alpha f(z)dz$ in the limit as $R \to \infty$:

$$\begin{aligned}\left|\int_\alpha f(z)dz\right| &= \left|\int_0^\pi \frac{e^{2\pi i\xi Re^{i\theta}}}{1+R^2e^{2i\theta}} iRe^{i\theta}d\theta\right| \\ &\leq R\pi \sup_{\theta\in[0,\pi]} \frac{e^{-2\pi\xi R\sin(\theta)}}{R^2-1}\end{aligned}$$

notice that for any value of $\theta \in [0,\pi]$, $\sin(\theta) \geq 0$ so the exponent must be negative, thus in the limit as $R \to \infty$ the expression will tend to zero and we see that

$$\int_\alpha f(z)dz \to 0 \qquad \text{as } R \to \infty$$

Thus in the limit as $R \to \infty$ we have

$$\int_\mathbb{R} f(z)dz = \pi e^{-2\pi\xi}$$

and taking the real part we see that

$$\int_\mathbb{R} \frac{\cos(2\pi x\xi)}{x^2+1}dx = \pi e^{-2\pi\xi}$$

as required.

□

(3) Does there exist a holomorphic surjection from the open unit disk D to the whole complex plane $\mathbb{C}$? If so, provide one; if not, prove that it does not exist.

Proof. First we use a biholomorphism $F : \mathbb{D} \to S$ where $S = \{(x,y) \mid x > -1\}$. The existence of such a biholomorphism is guaranteed by the Riemann mapping theorem. Now I claim that the following map $g : S \to \mathbb{C}$ defined by $z \mapsto z^2$ is surjective. Indeed any number in the closed right half plane can be written as $Re^{i\theta}$ with $\theta \in [-\pi/2, \pi/2]$ and so $g(Re^{i\theta}) = R^2e^{i2\theta}$ where now $R \in [0,\infty)$ and $\theta \in [-\pi,\pi]$, thus this map is surjective as the closed right half plane is a subset of S. It then follows that the composition $g \circ F : \mathbb{D} \to \mathbb{C}$ is a holomorphic surjection from the disk to the entire complex plane. □

(4) Let $\{z_1, \ldots, z_n\}$ be points on the unit circle in the complex plane. Prove that there exists a point z on the unit circle such that

$$1 = \prod_{k=1}^n |z - z_k|$$

Proof. Consider the function $f(z) = \prod_{k=1}^n (z - z_k)$, this is clearly holomorphic on an open set containing the closed unit disk. Notice that $|f(0)| = \prod_{k=1}^n |-z_k| = 1$ so by the maximum-modulus principle there exists a point $w \in \partial\mathbb{D}$ such that $|f(w)| \geq 1$. Now since S^1 is compact and $|f| : S^1 \to \mathbb{R}_{\geq 0}$ is continuous, $|f(w)| \geq 1$ and $|f(z_k)| = 0$ for any k, it follows from the intermediate value theorem that there exists a point $z \in \partial\mathbb{D}$ such that $|f(z)| = 1$ as required. □

(5) Let $A = \{1 < |z| < 2\}$ and $B = \{1 < |z| < 3\}$. Show that there is no holomorphic function $f : A \to B$ such that f extends continuously to the closure $\overline{A}$ to $\overline{B}$ and $f(\{|z| = 1\}) =\subset \{|z| = 1\}$ and $f(\{|z| = 2\}) \subset \{|z| = 3\}$.

Proof. Assume such a map f exists. Then notice that $\ln|f|$ is a harmonic function on A. Let

$$u(z) = \ln|f(z)| - \frac{\ln(3)}{\ln(2)} \ln|z|$$

and notice that $u(z) = 0$ when $|z| = 1$ and $|z| = 2$, so the maximum-modulus principle for harmonic functions tells us that $u(z) = 0$ on A. We then see that $|f(z)| = |z|^{\ln 3/\ln 2}$. Now for any $z_0 \in A$ we can find $\varepsilon > 0$ such that $f(z) = e^{i\theta} z^{\ln 3/\ln 2}$ (working on a suitable branch of the logarithm). Since we could do this for all $z \in A$ it follows that $\frac{f'(z)}{f(z)} = \frac{(\ln 3/\ln 2)}{z}$ for all $z \in A$. Then if we integrate both sides of the expression along a circle of radius 3/2

$$\int_\gamma \frac{f'(z)}{f(z)} dz = \int_\gamma \frac{(\ln 3/\ln 2)}{z} dz$$

we see that the right hand size is $2\pi i(\ln 3/\ln 2)$. However, I claim that the LHS is an integer multiple of $2\pi i$ which would yield a contradiction.

Replacing $f(z)$ with w we make a change of variables

$$\int_\gamma \frac{f'(z)}{f(z)} dz = \int_{f(\gamma)} \frac{1}{w} dw$$

By continuity, either $f(\gamma)$ is a point or a closed loop in the annulus and so $\int_{f(\gamma)} \frac{1}{w} dw = 2\pi n$ for some $n \in \mathbb{Z}$ and we have reached the desired contradiction. □