# Lattice Cohomology Talk

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The goal of part 1 is to give motivation and background regarding isolated singularities of complex algebraic curves and the relationship to knot theory and low dimensional topology. We will develop the necessary tools required for the lattice cohomology for isolated singularities, namely the semigroup of a branch associated to an isolated singularity of an algebraic curve. We will also define the delta invariant and study links of singularities.

**Isolated Singularities of Algebraic Curves** First we fix some notation:

- $\mathbb{C}[x, y]$ denotes the ring of polynomials over $\mathbb{C}$
- $\mathbb{C}[[x, y]]$ dentoes the ring of formal power series over $\mathbb{C}$
- $\mathbb{C}\{x, y\}$ denotes the subring of convergent power series over $\mathbb{C}$ in a neighborhood $U$ around $0$

**Theorem:** $\mathbb{C}\{x, y\}$ is a UFD

*Proof.* □

**Definition:** Let $f : \mathbb{C}^2 \to \mathbb{C}$ where $f \in \mathbb{C}[x, y]$, then we define $V(f)$ to be

$$V(f) = \{(x, y) \in \mathbb{C}^2 | f(x, y) = 0\}$$

that is the zero locus of $f$.

**Example:** Consider $f(x, y) = x^2 + y^2 - 1$: $V(f)$ is the following simply the set $(x, y)$ such that $x^2 + y^2 = 1$, that is *the unit circle*!
**Example:** Consider $f(x, y) = x^3 - y^2$: $V(f)$ is the following graph:
**Example:** Consider $f(x, y) = x^2 - y^2$: $V(f)$ is the cross

**Definition:** Let $f, g : U_1, U_2 \to \mathbb{C}$ be defined for some neighborhoods $U_1, U_2$ about $0$. $f, g$ define the same *germ* at $0$ if they are equivalent in some open subset $U \subset U_1 \cap U_2$ containing $0$. If $f, g$ are holomorphic, $f \sim g$ if and only if their power series expansions coincide.

Our considerations will be 'local' in the sense that we are interested in the behavior of our algebraic curve in the vicinity of an isolated singularity at 0. Thus we will implicitly always work with the germ defined by $f$. Of course I need to actually say what I mean by 'singularity' and 'isolated singularity'.

We define the Jacobian of $f$ as

$$df(\mathbf{z}) = \left(\frac{\partial f_i(\mathbf{z})}{\partial_j}\right)_{ij}$$

Definitions:

**Singular Locus:** We define the singular locus of $V(f)$ to be the subset:

$$\text{Sing } V(f) = \{x \in V(f) : \text{rank}(df(x)) < \text{rank}(df(x_{\text{generic}}))\}$$

moreover Sing $V(f) \subset V(f)$.

**Isolated Singularity:** $V(f)$ is said to have an isolated singularity at $0$ if for some $\varepsilon > 0$: Sing $V \cap B_\varepsilon = \{0\}$. So in a small neighborhood of the origin, $0$ is the only singular point, i.e. $0$ is the singular locus.

**Example:** If we consider the Morse function $x_1^2 + ... + x_n^2 = 0$ is singular on $\{0\}$ but $x_1^2(x_2^2 + x_3^2) = 0$ contains $\{x_1 = 0\}$ in its singular locus thus it is not an isolated singularity.

If $V$ is an isolated singularity then $V - \{0\}$ is a "smooth" complex manifold of (complex) dimension $\dim V$. This also carries a canonical orientation from the complex structure. Now we want to understand these objects from at least two points of view: the analytic and the topological.

**Analytic Type:** *Analytic type of* $(V, 0)$ *is the isomorphism type of* $(V, 0)$ *up to analytic isomorphism.*

Ideally we would like both topological and analytic invariants for our singularity. What we do here is to view $V(f)$ as an algebraic object, thus allowing us to use techniques from commutative algebra.

**Definition:** Let $\mathcal{O} = \mathbb{C}\{x, y\}$. For an (algebraic curve) variety $V(f)$ defined by $f(x, y) = 0$ we define the local ring $\mathcal{O}_{V(f)}$ by

$$\mathcal{O}_{V(f)} = \frac{\mathcal{O}}{\langle f \rangle} = \frac{\mathbb{C}\{x, y\}}{\langle f \rangle}$$

$\mathcal{O}_{V(f)}$ is a local ring in the sense that it has a unique maximal ideal: $m = \langle x, y \rangle$. This turns out to be the correct lense to analyze these singularities as the analytic type of $V$ is completely determined by the local ring $\mathcal{O}_{V,0}$ up to $\mathbb{C}$-algebra isomorphism!

Let $\gamma : \mathbb{C} \to \mathbb{C}^2$ be a parameterization of $f$. There are theorems determining the existence of "good" parameterizations. Moreover, locally the germ of $f$ is an element in $\mathbb{C}\{x, y\}$, thus since $\mathbb{C}\{x, y\}$ is a UFD, $f$ decomposes uniquely as:

$$f = \prod_i (g_i)^k$$

for irreducible $g_i$ for all $i$. Moreover, $V(g_i) \subset V(f)$ since if $g_i(x, y) = 0$, then $f(x, y) = g_i(x, y) \prod_{i \neq k} (g_k)^l = 0$. Each $g_i$ determines a *branch* of $f$. (later on we will see that each irreducible component also determines a component in the link of the singularity).

As an example if $f = x^2 - y^3 = 0$ in $\mathbb{C}^2, 0$ then $x(t) = t^3$ and $y(t) = t^2$ so the parameterization is given by local holomorphic functions.

Now suppose we have a parameterization $n : \mathbb{C} \to \mathbb{C}^2$, then this determines a ring homomorphism $\gamma^* : \mathbb{C}\{x, y\} \to \mathbb{C}\{t\}$. This is defined by $\sum_{i,j} a_{i,j} x^i y^j \mapsto \sum_{i,j} a_{i,j} x(t)^i y(t)^j = \sum_k a_k t^k$. The kernel of $\gamma^*$ is the ideal of functions which vanish on the branch $B$ defined by the parametrization (so $V(B) = \mathbb{C}\{x, y\} / \ker \gamma^*$). The quotient ring defined by the branch $\mathcal{O}_B$ is isomorphic (by first isomorphism theorem) to the image of $\gamma^*$ in $\mathbb{C}\{t\}$. Recall that the order of a polynomial (or power series) is the degree of the lowest degree nonzero term.

**Definition: Semigroup of a branch and the delta invariant**

Let $B$ be a branch, then the semigroup is defined as:

$$S(B) = \{\text{ord}\phi | \phi \in \mathcal{O}_B\}$$

The delta invariant is defined as

$$\delta = \dim \frac{\mathbb{C}\{t\}}{\mathcal{O}_{V,0}} = \#\{\mathbb{Z}_{\geq 0} - \varphi\}$$

The semigroup can easily be seen to be a subsemigroup of $\mathbb{Z}_{>0}$.

**Example:** Consider the curve defined by $x^3 = y^5$. Convince yourself that $t \mapsto (t^5, t^3)$ defines the parameterization. The semigroup is generated by $3, 5$ so $S(f) = \langle 3, 5\rangle$. $1, 2, 4$ are clearly in the complement, as is 7. We can generate $3+3 = 6$, $5+3 = 8$, $6+3 = 9$, $5+5 = 10$, $8+3 = 11$, $9+3 = 12$, and from here we see that we can generate every greater element. Thus $\{1, 2, 4, 7\}$ are all elements in the complement and $\delta(f) = 4$.
**Exercise:** $x^a + y^b$ $\varphi = \langle a, b\rangle$ with $a$ and $b$ relatively prime, then $\delta = \frac{(a-1)(b-1)}{2}$

We will now turn to studying the "link" of an isolated singularity. As part of our definition of "isolated" we know that $V(f)$ has a full rank jacobian, and thus by the implicit function theorem, $V(f) - 0$ forms a complex manifold of complex dimension 1. Moreover, we have the following result: there exists $\varepsilon_0 > 0$ such that $S^{2n-1}$ (a sphere bounding a ball of radius $\varepsilon$ about the origin) intersects $V(f)$ transversely. Moreover, $\forall \varepsilon$ $0 < \varepsilon < \varepsilon_0$, $V(f) \cap S^{2n-1}$ have the same topological type. This forms the link:

**Definition:** Let $V(f)$ have an isolated singularity at $0$.

$$\text{Link}(V(f)) = V(f) \cap S^{2n-1}_{\varepsilon_0}$$

What is this "link" topologically? Recall from the previous paragraph that $f$ uniquely factorizes as a product of irreducible components. As $V(g_i) \subset V(f)$ it is not hard to reason that Link$(g_i) \subset$ Link$(f)$, and moreover, $g_i \cap S^{2n-1}$ is precisely one component, which by transversality must have real dimension 1, and moreover is closed; thus Link$(g_i) \cong S^1$. Moreover, Link$(V(f)) = \sqcup_{i=1}^k S^1$. By transversality and the fact that $V(f) - 0$ in particular is a smooth manifold, it follows that the canonical embedding of the link into $S^{2n-1}_{\varepsilon_0}$ is smooth, thus it indeed forms a link in the usual knot theory sense. This is an important part in the history of knot theory as these 'algebraic links' were among the first to be seriously studied.

If we work in a dimension higher and consider $f : \mathbb{C}^3 \to \mathbb{C}$, then the link will be the transverse intersection of a complex plane which has real dimension 4 with a 5 dimensional sphere, which then by transversality means that the link has dimension $n = 4 + 5 - 6 = 3$. It will turn out that links of surface singularities are in fact plumbed three manifolds with negative definite intersection form.

# I Construction of the Lattice Cohomology

Our goal will be to assign from a geometric object some data from which we get a cohomology theory.

Geometric object $\to$ Data $\to$ cohomology

$$\mathbb{H}^* = \bigoplus_{q \geq 0} \mathbb{H}^q$$

and $\mathbb{H}^q$ is a $\mathbb{Z}$-graded $\mathbb{Z}[u]$-module $(\deg u = -2)$. The nice part is that lattice cohomology can be defined purely combinatorialy, thus Data $\to$ cohomology is relatively easy. The hard part is Geometric object $\to$

Data that is generating the relevant data from our geometric/topological objects which forms a cohomology that 1) is an invariant (topological, smooth, analytic) and 2) contains useful information. Several lattice cohomologies have been developed for objects arising in singularity theory and low dimensional topology.

## I.1 Data

There are 3 pieces of data associated to our geometric object, a $\mathbb{Z}^r$-lattice, an ordered basis $\{E_i\}_{i=1}^r$ on which we can define a partial ordering on the lattice and a cubical decomposition of $\mathbb{R}^r$. The 0-cubes are $l \in \mathbb{Z}^r$ and 1-cubes are the edges $[l, l+E_i]$ for $l \in \mathbb{Z}^r$. A 2-cube $[l, E_i, l+E_j, l+E_i+E_j]$, and so on. As a remark $\mathbb{Z}^r \rightsquigarrow (\mathbb{Z}_{\geq 0})^r \rightsquigarrow R(0,c)$ where $c \in \mathbb{Z}_{\geq 0}$. Lastly we need a weightfunction on the lattice. $w : \mathbb{Z}^r \to \mathbb{Z}$ with the condition that $\#w(-\infty, n] < \infty$. Essentially this means that there exists a minimum weight, $n_w$. This weight then extends to the $q$-cubes which I will denote $\square^q$ where $q$ is the dimension of the cube. The weight of the cube $w(\square^q) = \max\{w(l), l \text{ is vertex of } \square^1\}$. Now $S_n = \cup_{w(\square) \leq n} \square$. One can easily see that this forms a filtration of $\mathbb{Z}^l$

$$\emptyset \hookrightarrow S_{w_m} \hookrightarrow \ldots \hookrightarrow S_n \hookrightarrow S_{n+1} \hookrightarrow \ldots$$

## I.2 Cohomology

The cohomology is the following $\mathbb{Z}[u]$ module formed by previous filtration:

$$\mathbb{H}^q = \bigoplus_{n \geq m_w} \mathbb{H}^q(S_n; \mathbb{Z})$$

where $n$ gives a $\mathbb{Z}$-grading and $\mathbb{H}^q(S_n; \mathbb{Z})$ is the qth singular cohomology of $S_n$. We are now able to define the $u$ action turning this into a $\mathbb{Z}[u]$ module. The $u$ action $u : H^q(S_n; \mathbb{Z}) \to H^1(S_{n-1}, \mathbb{Z})$ is simply the action by restricting the domain from $S_{n+1}$ to $S_n$. We also require $\deg u = -2$ (for historical reasons coming from Heegaard-Floer).

**Example 0:** Consider $\mathbb{Z}_{\geq 0}$ as the lattice. Let $w(x) = x$. It is clear that there is no nontrivial homology for $q > 1$ and $\mathbb{H}^0(S_n; \mathbb{Z}) \cong \mathbb{Z}$ for all $n$. Thus we obtain a $\mathbb{Z}$ tower $\mathcal{T}$. Moreover $\mathbb{H}^0_{red} = 0$.

Moreover if $w(l + E_i) \geq w(l)$ for all $l, i$ then $S_n$ is contractible and the cohomology is trivial.

**Example 1:** In this example we will consider the lattice cohomology defined by a semigroup in $\mathbb{Z}_{\geq 0}$. Let us assume $0 \in \varphi$ so it forms a monoid. Assume that $\delta(\varphi) < \infty$ (so there are only finitely many gaps). Define

$$h(\ell) = \#\{s \in \varphi, s < \ell\}$$

so $h$ looks at the number of semigroup elements below each element. The lattice cohomology of $h(l)$ is trivial since its increasing, so we define

$$\bar{h}(\ell) = \#\{s \in \mathbb{Z}_{\geq 0} - \varphi, g < l\}$$

counts the number of gaps below. It's increasing, hits $\delta$ and then remains constant.

$$w(\ell) = h(\ell) - \bar{h}(\ell)$$

This measures the distribution of the semigroup.

So to any semigroup we have a lattice cohomology.

**Example 2:** Use the projector.

**Euler Characteristic** Assume that $\text{rank}_{\mathbb{Z}} \mathbb{H}^q_{red} < \infty$, then

$$\chi(\mathbb{H}^*) = -m_w + \sum_q (-1)^q \text{rank}(\mathbb{H}^q_{red}) = \sum_{\square_q} (-1)^{q+1} w(\square_q)$$

We have given an analytic lattice cohomology for the singularity but what can we do for the topological side? On its own the link topologically is just a bunch of $S^1$'s so not so interesting; however, we can think of $V$ as a plane curve, thus the link can be embedded into $S^3$ which has a rich topology (knot theory). Here we have the link Floer homology $HFL^{-}(L)$. This categorifies the multivariate Alexander polynomial which corresponds on the analytic side to the Hilbert function. The Hilbert function from $\varphi$ is:

$$H(t) = \sum_{\ell \geq 0} h(\ell) t^{\ell}$$

Consider $x^3 = y^2$, then $t \mapsto (t^2, t^3)$ is the parameterization which gives rise to $\varphi = \langle 2, 3 \rangle$ as the semigroup. We should note that the topological type of the link of $x^2 = y^3$ is the trefoil (2,3) torus knot. Taking the Hilbert function we see:

$$\begin{aligned}\sum_{s \in \varphi} t^s =& 1 + t^2 + t^3 + t^4 + .... \\ =& 1 + t^2 \frac{1}{1-t} \\ =& \frac{1 - t + t^2}{1-t}\end{aligned}$$

which is the Alexander polynomial of $3_1$!

Now if our curve is not irreducible, then

$$(C, 0) = (C_1, 0) \cup .... \cup (C_r, 0)(\mathbb{C}^2, 0)$$

then the Hilbert function:

$$H(t_1, ...t_r) = \sum_{\ell \in (\mathbb{Z}_{\geq 0})^r} h(\ell) t_1^{\ell_1} ... t_r^{\ell_r}$$

We are going to get the multivariable Alexander polynomial:

Now if we define the weight function

$$w(\ell) = 2h(\ell) - \ell$$

then we get a Lattice cohomology for the topological type of $V(f)$.

**Example:** $V(f) = \{x^2 - y^4\} = \{(x - y^2)(x + y^2)\}$ which has two irreducible components. Thus Link$(V) = S^1 \sqcup S^1$.

$$\begin{aligned}P_{L_1}(t_1) =& \frac{1}{1-t_1} \\ P_{L_2}(t_2) =& \frac{1}{1-t_2}\end{aligned}$$

So

$$P_L(t_1, t_2) = 1 + t_1 t_2$$

## II Surface Singularities

If $\dim_{\mathbb{C}} V = 2$ (normal surface singularity) then $\dim_{\mathbb{R}}$ link$(V) = 3$. The topological type of the link for isolated singularities is a negative definite plumbed 3-manifold. Moreover, we can compute its Heegaard Floer homology $HF^{-}$ whose Euler characteristic is the Seiberg-Witten invariant. We can also compute a lattice cohomology: $\mathbb{H}^*$ whose Euler characteristic also yields $SW$.

When Link$(V)$ is a $\mathbb{Q}HS^3$ then

$$\mathbb{H}^* \cong HF^-[Link]$$

On the analytic side, there is $\mathbb{H}^*_{an}$ whose Euler characteristic yields the $Pg$ (geometric genus). Between $\mathbb{H}^*_{an}$ and $\mathbb{H}^*_{top}$ there is a functor: $\mathbb{H}^*_{an} \to \mathbb{H}^*_{top}$. Our goal is to classify analytic types for a topological type. Since we mentioned functors, we need to specify a category. On the topological side the morphisms are cobordisms, while on the analytic side the morphisms are deformations of singularities.