Knot Concordance Notes

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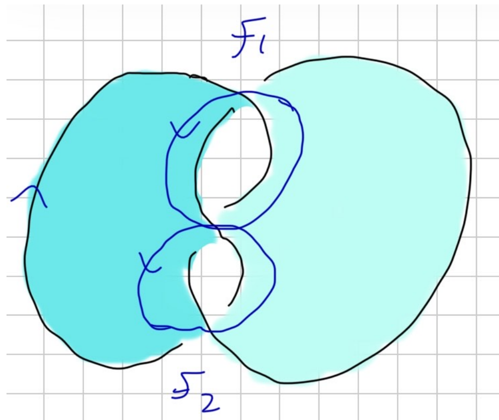
Introduction These are some personal notes to help make things precise in my mind. There are 3 notions of concordance (actually 4 if we include homology concordance) in the literature that I found which are in general *not* equivalent. These are algebraic, topological, and smooth concordance. To each there is an associated group called the (top, sm, alg) concordance group which will be denoted $\mathscr{C}^{top}, \mathscr{C}^{sm}, \mathscr{C}^{alg}$. A main result is that there exists group homomorphisms that surject as follows

$$\mathscr{C}^{sm} \overset{\text{onto}}{\to} \mathscr{C}^{top} \overset{\text{onto}}{\to} \mathscr{C}^{alg}$$

It is known that $\mathscr{C}^{alg} \cong \mathbb{Z} \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$, but $\mathscr{C}^{top}$ and $\mathscr{C}^{sm}$ are not known up to isomorphism. It remains to fully capture torsion subgroups in $\mathscr{C}$, see [6]. The existence of knots which are topologically slice but not smoothly slice can be used to prove the existence of exotic smooth structures on $\mathbb{R}^4$. There is some effort in trying to accomplish similar goals for $\mathbb{S}^4$ in hopes of disproving the Poincare conjecture, but this remains unresolved.

I.1 Preliminaries: Seifert form

Let K, $S^1 \hookrightarrow S^3$ be a knot in S^3. To this knot we can construct oriented compact surfaces whose boundary is K. Such a surface is called a Seifert surface. Given a Seifert surface F, A Seifert form is a bilinear form which takes pairs of classes in $H_1(F)$ and returns the linking number of a loop in $[f]$ and the positive pushoff, $\iota_+\beta$. Such a form can be represented by a matrix of dimension $(2g \times 2g)$, where g is the knot genus. Seifert forms for a knot K admit different matrix representations up to S-equivalence. The Alexander polynomial can be computed as $\det(tA - A^T)$, while the signature $\sigma(K) = (1-\omega)A + (1-\bar{\omega})A^T$, where ω is unit modulus complex number not equal to one.
Example: Consider the following Seifert surface of the Trefoil.



We can now find the entries of the Seifert matrix.

$$\begin{aligned} lk(f_1, f_1^+) =& 1 \\ lk(f_1, f_2^+) =& -1 \\ lk(f_2, f_1^+) =& 0 \\ lk(f_2, f_2^+) =& 1 \end{aligned}$$

So A,

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

We can compute the Alexander Polynomial easily as $\Delta_{3_1}(t) = t^2 - t + 1$, which can be normalized by t^{-1} as $\Delta_{3_1}(t) = t + t^{-1} - 1$. The signature can also be easily computed: $\sigma(3_1) = -2$

Theorem: *Let A be a square matrix with integer coefficients. A is a Seifert matrix if and only if* $\det(A - A^T) = \pm 1$.

Proof. $\Rightarrow$ The forward direction comes as a corollary to Lickorish theorem 6.10 (ii) [4]. An outline of the arguement is that entries in $A - A^T$ yield the intersection form on F, which has determinant plus or minus one.

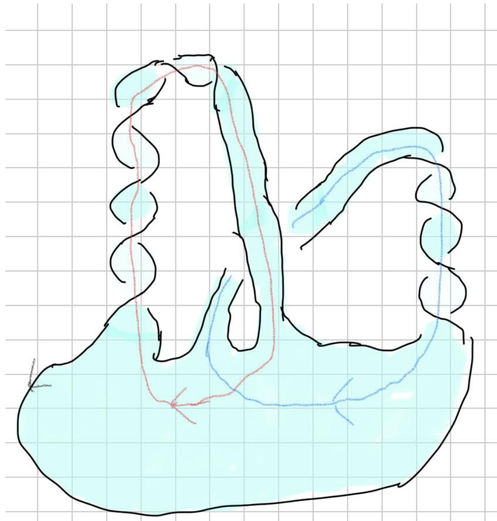
$\Leftarrow$ Notice that $A - A^T$ is anti-symmetric, so in order for $\det(A - A^T)$ to be nonzero, A must be even-dimensional. Now we can build a Seifert surface of genus g by glueing (g) pairs of "bands" (as in figure 6.1) and twisting and linking them properly to obtain the coefficients in A. ☐

Corollary: Exercise 6.7 in Lickorish. Suppose B is any $2n \times 2n$ matrix of integers with the property that $B - B^T$ consists of n blocks of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ running down the diagonal and zeros elsewhere. Then there exists a knot for which B is a Seifert matrix.

Example: Suppose we are given the following Seifert matrix A,

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

We verify that this is in fact a Seifert matrix: $\det(A - A^T) = -1$. Now the dimension tells us that F has 2 bands that generate the homology. This is realized in the following surface



Proposition: If A_{F_1} and A_{F_2} are Seifert matrices for F_1 and F_2, then $A_{F_1 \# F_2} = A_{F_1} \oplus A_{F_2}$.

Proof. Consider a Mayer-Vietoris exact sequence, where $A = F_1$, $A = F_2$ and $A \cap B = I \times I$, it follows that since all these spaces are path connected, that

$$H_1(F_1 \# F_2) \cong \frac{H_1(F_1) \oplus H_1(F_2)}{\text{im } \phi}$$

where $\phi : H_1(F_1 \cap F_2) \to H_1(F_1) \oplus H_1(F_2)$ is the homomorphism in the exact sequence, but since $H_1(F_1 \cap F_2) = H_1(I \times I) = 0$, it follows that $H_1(F_1 \# F_2) \cong H_1(F_1) \oplus H_1(F_2)$, and we conclude that $A_{F_1 \# F_2} = A_{F_1} \oplus A_{F_2}$. □

I.2 Sliceness & Concordance

Suppose a knot K bounds a flat embedded disk $\mathbb{D}^2$ in S^3 then we know that it is the unknot. What can we say however about a knot which bounds a disk in B^4? Stated this way we can see that the answer is trivial since any knot K bounds a disk in B^4; simply take the cone CK which we can embed into $S^3 \times [0, 1]$ and moreover since $K \cong_{homeo} S^1$, it follows CK is homeomorphic to a disk. The cone of a knot however has a singularity at the apex which makes the disk not a particularly nice embedding. We can reformulate our question into posing whether or not a knot bounds a locally flat disk, which leads us to the following definition.

Topologically Slice *A knot K, $S^1 \hookrightarrow S^3$ is said to be topologically slice, if it bounds a locally flat embedding of a disk D^2 in B^4, such that $\partial(B^4, D^2) = (S^3, K)$. Such a disk is said to be a slice disk.*

We should be precise in what we mean by "locally flat". This means that $D \subset B^4$ has a neighborhood N such that $N(D) = D \times I^2$ which meets S^3 in $\partial D \times I^2$ (which on the boundary of B^4 will be a regular neighborhood of the knot i.e. a solid torus). Local flatness ensures that we avoid trivialities such as taking a cone over a knot, and also ensures that the slice disk avoids singularities. We also need to emphasize that although in S^3, the theory of knots is equivalent for locally flat embeddings (or piecewise linear embeddings) and smooth embeddings, w.r.t. 4 dimensional topology, the smooth and topological categories are not equivalent. It should be emphasized that being topologically slice does not necessary imply being smoothly slice, i.e. that the slice disk is a smoothly embedded submanifold.

Smoothly Slice A knot $K, S^1 \hookrightarrow S^3$ is said to be smoothly slice if it bounds a smoothly embedded disk D^2 in B^4. Two knots K_1 and K_2 are said to be concordant if there exists a smoothly embedded cylinder $S^1 \times [0, 1]$ in $S^3 \times [0, 1]$ whose boundary is $K_1 \sqcup -\bar{K}_2$.

Example: 6_1 is an example of a slice knot. The way to see this is by looking at a "slice movie". The disk will at every slide of the movie look like a finite number of closed loops starting at the first frame with the knot (so we have one loop), then almost all frames are simply isotopies of K except at some finite times, we meet a "saddle" point where we switch crossings and the succesive time steps will see an increase in the number of unknotted link components. We can then think of this movie combinatorially as a finite sequence of isotopies and moves whereby two strands are brought close together and switched (see the example below).

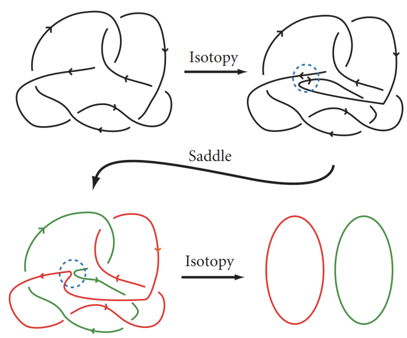


Figure 1: Source Julia Collins: University of Edinburgh [1]

There is a very famous example of a knot that is topologically slice but *not* smoothly slice, the Conway knot! It was known for a long time that the Conway knot was topologically slice, but only fairly recently was it proven by Lisa Picarillo that it is in fact not smoothly slice.

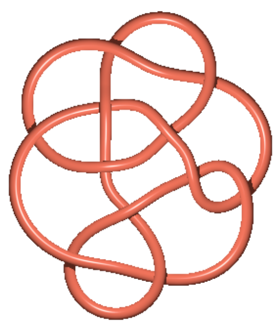


Figure 2: Conway Knot

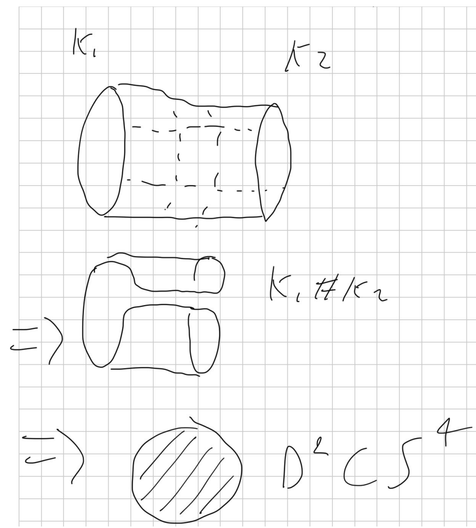
Picarillo's proof requires the Rasmussen invariant and Khovanov homology, and is outside the scope of these notes, but is certainly something I plan to revisit in the future. Next we discuss concordance, and its relation to slice knots.

Proposition: *Concordance is an equivalence relation on knots.*

Proof. Let K, $S^1 \hookrightarrow S^3$ be a knot, then if we simply take $K \times [0,1]$, where $K \times 0 = K \subset S^3$, then $\partial K \times [0,1] = K \sqcup -\bar{K}$, thus concordance is reflexive. Let K_1 and K_2 be concordant. If we simply reverse the concordance cylinder, then we obtain a concordance from K_2 to K_1. Transitivity follows from gluing together cylinders. □

Proposition: *For any knot K, $K\# -\bar{K}$ is smoothly slice. Moreover if K_1 and K_2 are concordant then $K_1\# -\bar{K}_2$ is slice.*

Proof. First, I claim that if two knots K_1 and K_2 are concordant, then $K_1\#K_2$ is slice. Observe the following diagram,



We take K_1 and K_2 to be concordant, and observe this cylinder between them which is understood to be PL (or smoothly) embedded in $\mathbb{S}^3 \times [0,1]$, if we join these two knots, what we are doing effectively to the cylinder is cutting along its vertical axis and forming a disc $\mathbb{D}^2$ which satisfies the flatness property necessary to be a slice disk (or smoothness property in the case of smoothly concordant), It follows immediately that since $K_1\#K_2$ admit a slice disk, that $K_1\#K_2$ is slice. Moreover since any knot K is concordant to itself, it follows that $K_1\# - \bar{K}_2$ is slice. $\square$

We can also see that if $K \sim_{conc} 0_1$, then K is slice. (simply cap off the unknot in the cylinder), so the $[0_1]$ class is precisely the class of (sm or top) slice knots.

Proposition: (sm or top) Concordant knots have vanishing signature.

Proof. From the previous proposition, if two knots K_1 and K_2 are concordant, then $K_1\#K_2$ is slice.

Next, I claim that for two knots K_1 and K_2, $\sigma(K_1\#K_2) = \sigma(K_1) + \sigma(K_2)$. Suppose A_{K_1} and A_{K_2} are Seifert matrices for K_1 and K_2 respectively, then

$$A_{K_1\#K_2} = \begin{pmatrix} A_{K_1} & 0 \\ 0 & A_{K_2} \end{pmatrix}$$

. It follows that

$$(1-\omega)A_{K_1\#K_2} + (1-\bar{\omega})A_{K_1\#K_2} = \begin{pmatrix} (1-\omega)A_{K_1} + (1-\bar{\omega})A_{K_1} & 0 \\ 0 & (1-\omega)A_{K_2} + (1-\bar{\omega})A_{K_2} \end{pmatrix}$$

Thus it follows that $\sigma(A_{K_1\#K_2}) = \sigma(K_1) + \sigma(K_2)$.

It follows now that if K_1 and K_2 are concordant, then that since their join is slice and slice knots have signature zero, then $\sigma(K_1\#K_2) = 0$ implies $\sigma(K_1) = \sigma(K_2)$ as required. $\square$

While knot signature can be used to obstruct knots from being (sm, top, alg) slice, it cannot distinguish between those three categories. Stronger invariants are required to distinguish between them.

I.3 Concordance Groups

In this section, we show that the equivalence class of knots under concordance together with the connect sum operation on knots has the structure of an Abelian group. First we need to show that the connect sum respects concordance classes:

Proposition: (well definedness) If $K_1 \sim K_2$ and $J_1 \sim J_2$, then $K_1 \# J_1 \sim K_2 \# J_2$.

Sketch of Proof. If we take concordances C_1 and C_2, and cut the cylinder via cutting the knot and glue the boundaries via the join, then we obtain a concordance of $K_1 \# J_1$ and $K_2 \# J_2$.

Theorem: The set $\mathscr{C}$ of concordance classes form an Abelian group under the connect sum operation of knots. The class of 0_1 which consists of all slice knots form the identity element. Inverses of $[K]$ are precisely $[-\bar{K}]$.

Proof. From the previous proposition, we know that this is well defined. It inherits associativity and commutativity from connect sum of knots. Lastly since we know that $K \# -\bar{K}$ is slice, it follows that the inverse of $[K]$ is precisely $[-\bar{K}]$. □

We can do the arguments in the previous theorem and proposition for both the smooth and top case. We can also construct a natural surjective homomorphism $\phi : \mathscr{C}^{sm} \to \mathscr{C}^{top}$ sending a class in $\mathscr{C}^{sm}$ to its class in $\mathscr{C}^{top}$, this is indeed well defined as any two smoothly concordant knots are certainly topologically concordant. Moreover we can view $\mathscr{C}^{top}$ as being a subgroup of $\mathscr{C}^{sm}$.

I.3.1 Properties of $\mathcal{C}^{top,sm}$

Consider the figure eight knot (4_1), recall that it is negatively amphichiral, meaning that it is equivalent to its mirror and equivalent to its reverse. It follows then that in the concordance group, $[4_1]$ has order 2. Moreover, one can show that there are infinitely such summands of order two which live in the concordance group, giving us $\bigoplus_{n=1}^{\infty} \mathbb{Z}_2$.

There are also summands $\bigoplus_{n=1}^{\infty} \mathbb{Z}$. In the next section on algebraic concordance we will prove that that knots with non-vanishing signature have infinite order in $\mathscr{C}^{alg}$. Since an obstruction to algebraic sliceness is an obstruction to top and smooth sliceness, we can see that this argument extends to the topological and smooth categories, and it follows $[3_1]$ is of infinite order in $\mathscr{C}^{top,sm}$. Vanishing signature however is not a sufficient condition to determine finite order in the concordance groups. There are classes of finite order knots in $\mathscr{C}^{alg}$ which have infinite order in $\mathscr{C}^{sm}$. It is still an open problem whether there exists any other torsion in $\mathscr{C}^{top,sm}$.

Theorem: A knot with trivial Alexander polynomial is topologically slice.

I.4 Algebraic Concordance

Algebraic concordance is a way to capture the algebraic essence of concordance. The Seifert form of a topologically slice knot satisfies a "half lives half dies" argument whereby half of the generators of $H_1(F)$ for K are sent to zero by the Seifert form which in the matrix representation takes the form of a matrix with a block of zeros in the upper left. (for a proof see [4] proposition 8.17). A matrix in that form is called metabolic, and algebraic concordance is an equivalence relation on the set of Seifert matrices (which we defined in the preliminaries) such that the zero class is precisely the set of metabolic matrices. In this way $\mathscr{C}^{alg}$ reflects the structure concordance and sits as a subgroup in $\mathscr{C}^{sm,top}$.

Definition: Metabolic Form *A bilinear form $\alpha : V \to \mathbb{R}$ is said to be metabolic if $V = V_1 \oplus V_2$, and for any two $v, w \in V_1$, $\alpha(v, w) = 0$.*

Equivalently the matrix form A of α is metabolic if for some $U \in GL_n(\mathbb{Z})$ (unimodular matrices) A is of the following form

$$UAU^T = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$$

Lickorish proved in chapter 8 that topologically slice knots admit metabolic Seifert matrices. However, admitting metabolic Seifert matrices does not characterize (topologically or smoothly) slice knots.

Proposition: If K is algebraically slice, then the Conway-normalized Alexander polynomial of K is of form $f(t)f(t^{-1})$ (this is known as the Fox-Milnor condition), where f is a polynomial with integer coefficients. Additionally, $\sigma(K)$ vanishes.

Proof. The proof is essentially equivalent to the proof of Theorem 8.18 in Lickorish, except we note that this theorem generalizes to algebraically slice. For the second part, see the proof of Theorem 8.19 in [4] □

The next goal will be to construct the algebraic concordance group, by modding out the Seifert matrices by Metabolic forms. The following series of theorems and propositions are taken from Conway [2]
Lemma [2]: *Given Seifert matrices A, B, A', B', M_1, M_2 and M., the following statements hold.*
(1) If M_1 and M_2 are metabolic, then $M_1 \oplus M_2$ is metabolic
(2) If $A \oplus -A'$ and $B \oplus -B'$ are metabolic, then $(A \oplus B) \oplus -(A' \oplus B')$ is metabolic;
(3) If M and $A \oplus M$ are metabolic, then A is metabolic.

Proof. Let $P_1, P_2 \in Gl_n(\mathbb{Z})$ be (2g and 2h dimensional respectively) such that $P_1 M_1 P_1^T$ and $P_2 M_2 P_2^T$ have g and h dimensional blocks of zero in their upper left corner. Let P be the $2(g+h)$ square permutation matrix which places the aforementioned blocks of zeros next to each other in the basis. Clearly, $P(P_1 \oplus P_2)(M_1 \oplus M_2)(P_1 \oplus P_2)^T P^T$ contains a $g+h$ block of zeros in its upper left hand corner.
To prove (2) Let P be the permutation matrix s.t. $P((A \oplus B) \oplus (-(A' \oplus B')))P^T = (A \oplus A') \oplus (B \oplus B')$, and it follows from (1) that this is metabolic.
(3) This last point is knot as *Witt cancellation* and is proven in Levine [3]. □

Definition: *A knot K is said to be algebraically slice if it admits a metabolic Seifert form. Two abstract Seifert forms are said to be* algebraically concordant *if $V_1 \oplus -V_2$ is metabolic.*

Theorem: [2] *Algebraic concordance is an equivalence relation on the set $\mathscr{S}$ of Seifert matrices. Moreover, the set $\mathscr{C}^{alg}$ of algebraic concordance classes forms an abelian group called the algebraic concordance group: the group law is induced by the direct sum, the zero element is represented by metabolic matrices and the inverse of a class $[A]$ is $-[A]$.*

Proof. Reflexivity: Let A be a $2n \times 2n$ Seifert matrix, and consider $A \oplus -A$. Let P be the matrix which adds the $2m+i$th row to the ith row, for $4m \times 4m$ matrix, in block form $P = \begin{pmatrix} I_{n\times n} & I_{n\times n} \\ 0 & I_{n\times n} \end{pmatrix}$. Moreover,

$$\begin{pmatrix} I_{n\times n} & I_{n\times n} \\ 0 & I_{n\times n} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \begin{pmatrix} I_{n\times n} & I_{n\times n} \\ 0 & I_{n\times n} \end{pmatrix}^T = \begin{pmatrix} 0 & -A \\ -A & -A \end{pmatrix}$$

Thus $A \oplus -A$ is metabolic.
Symmetry: Suppose $A \sim B$, where A is $m \times m$ and B is $n \times n$. Suppose U is such that $U(A \oplus -B)U^T$ has $m+n$ block of zeros in the upper left, Now simply interchange the first m rows of U with the next n rows to form Q, and it follows that $Q(B \oplus -A)Q^T$ has $m+n$ block of zeros in the upper left.
Transitivity: Suppose $A \sim B$ and $B \sim C$, then $(A \oplus -B) \oplus (B \oplus -C) = (A \oplus -C) \oplus (B \oplus -B)$, and since $(B \oplus -B)$ is metabolic, and by (2), and then (3) of the earlier lemma, it follows that $a \oplus -C$ is metabolic.

Abelian Group: (2) of the previous lemma demonstrates algebraic concordance under direct sum as a well defined group operation. The identity is clearly the class of metabolic Seifert matrices. Inverses of a class $[A]$ is $[-A]$. Associativity and Commutativity are also clear.

□

Theorem: [2] *The map that sends the concordance class of a knot to the algebraic concordance class of any of its Seifert matrices gives rise to a well-defined group homomorphism $\phi : \mathscr{C}^{sm} \to \mathscr{C}^{alg}$*

Levine was able to prove that

$$\mathscr{C}^{alg} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty \tag{1}$$

Proposition (Exercise 3.2.7 [5]): Show that a quadratic Alexander polynomial for a slice knot has the form $nt^2 - (2n+1)t + n$, where $n = k(k-1)$, $k \in \mathbb{N}$.

Proof. Let A be a Seifert matrix for such a knot, then it follows that since $\Delta_K(t)$ is quadratic, K is slice, and that A is 2×2, and moreover since $\det(A - A^T) = \pm 1$, that A takes the form

$$A = \begin{pmatrix} 0 & k \\ k-1 & l \end{pmatrix}$$

then

$$\begin{aligned} \Delta_K(t) =& -(tk - (k-1))(t(k-1) - k) \\ \doteq& t^2 k(k-1) - (k^2 + (k-1)^2)t + k(k-1) \\ =& t^2 k(k-1) - (2k(k-1)+1)t + k(k-1) \end{aligned}$$

Let $n = k(k-1)$,

$$\Delta_K(t) = nt^2 - (2n+1)t + n.$$

□

Proposition (Exercise 3.2.12 [5]): Show that if a knot K has finite concordance order, i.e., if there is a positive integer n such that nK is slice, then $\sigma(K) = 0$.

Proof. Recall that if a knot K is slice, then $\sigma(K) = 0$, and recall that for knots K and J, $\sigma(K + J) = \sigma(K) + \sigma(J)$. Now suppose that K has finite concordance order n, so nK is slice, then

$$\begin{aligned} \sigma(nK) =& n\sigma(K) \\ 0 =& n\sigma(K) \end{aligned}$$

Thus $\sigma(K) = 0$. □

It follows immediately that the trefoil has infinite order in the concordance groups since its signature is non-vanishing.

We should at this point note that all invariants for concordance and invariants detecting sliceness are algebraic concordance invariants and cannot effectively provide obstructions between $\mathscr{C}^{alg,sm,top}$. Though some, as an example trivial Alexander polynomial, are indeed strong enough to detect topological sliceness, generally since these invariants are defined from the Seifert form, they are capped as being invariants of $\mathscr{C}^{alg}$ which is the coursest of these concordance groups. The first obstructions to $\mathscr{C}^{top,sm}$ for algebraically slice knots are the Casson-Gordon invariants which will be disgussed in a later section.

I.4.1 Proof of $\mathscr{C}^{alg}$ Isomorphism

Definition: $\mathscr{C}^{alg}_{\mathbb{F}}$ *is the algebraic concordance of Seifert matrices over a field* $\mathbb{F}$ *where the Seifert matrices are defined as satisfying the following condition:*

$$\det((A - A^T)(A + A^T)) \neq 0$$

The first goal here is to establish the isomorphism between $\mathscr{C}^{alg}_{\mathbb{Z}}$ and $\mathscr{C}^{alg}_{\mathbb{Q}}$. The first thing we should notice is that there is a natural map ϕ from $\mathscr{C}^{alg}_{\mathbb{Z}}$ to $\mathscr{C}^{alg}_{\mathbb{Q}}$ which takes a representative from a concordance class over $\mathbb{Z}$ and sends it to the class over $\mathbb{Q}$. While it differs from the usual spirit of calling a map an inclusion map, it will turn out to be an injective map, so we will call this the *natural inclusion map*. Our goal will be to show that ϕ is an injective homomorphism, with the discussion of surjectivity following later.
Proposition: ϕ is a homomorphism

Proof. Let $[A]$ and $[B]$ denote two classes in $\mathscr{C}^{alg}_{\mathbb{Z}}$, and let $[C] = [A] * [B]$, then $\phi([A] * [B]) = \phi([C])$, and I claim that $\phi([C]) = \phi([A])\phi([B])$. This holds simply from the fact that we can choose representatives that $\phi([C])$ has representative C, which has an integral metabolizer that corresponds to $[C] \in \mathscr{C}^{alg}_{\mathbb{Z}}$. □

To prove injectivity, we require the following lemma.
Lemma: Every concordance class in $\mathbb{C}^{alg}_{\mathbb{Z},\mathbb{F}}$ has a non-singular representative.

Sketch of Proof The idea of the proof is to show that given a Seifert matrix A (over $\mathbb{Z}$ or $\mathbb{F}$ is concordant to an elementary reduction B (of the form in Lickorish), and since we may do elementary reductions until we obtain a non-singular matrix, the lemma follows.
Proposition: ϕ is an injective map.

Proof. Since ϕ is injective, it suffices to show it has trivial kernel. Let A be a non-singular representative from the 0 class in $\mathbb{C}^{alg}_{\mathbb{Q}}$. Let H be a rational metabolizer for A, it remains to show that A has an integral metabolizer. Viewing $\mathbb{Z}^{2n} \subset \mathbb{Q}^{2n}$, let $\mathbb{Z}^{2n} = V$, let $H_0 = H \cap V$. Since A vanishes on $H \times H$, it vanishes on $H_0 \times H_0$.

Suppose $\{h_1, ..., h_n\}$ is a basis for H over $\mathbb{Q}$. Since $h_i = (p_i^1/q_i^1, ..., p_i^{2n}/q_i^{2n})$, let $a_i = LCM(q_i^1, ..., q_i^{2n})$, then $a_i h_i \in H_0$ form a linearly independent basis of H over $\mathbb{Z}$ with rank n, and moreover as $H_0 \otimes_{\mathbb{Z}} \mathbb{Q} \subset H \cong \mathbb{Q}^n$, the rank of H_0 is n. It remains to show that H_0 is a direct summand. Let $v \in V$ and suppose for some non-zero integer m, $mv \in H_0 \subset H$. Since H is a rational vector space, $v = (1/m)mv \in H$ which implies that $v \in H \cap V = H_0$. It follows that V/H_0 is torsion-free and thus the exact sequence

$$0 \to H_0 \to V \to V/H_0 \to 0$$

is split exact, thus H_0 is a direct summand of $\mathbb{Z}^{2n}$. □

Surjectivity will follow after we construct infinitely many copies of $\mathbb{Z}, \mathbb{Z}_2$ and $\mathbb{Z}_4$.

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